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The internal energy of ferroelectric domain structures characterized by pre-fractals of the pentad Cantor sets

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Abstract. The laminated ferroelectric 180° domain structure optically observed in uniaxial ferroelectric KDP is periodic along the normal to the domain walls. Each half-period of the domain structure is characterized by the first four pre-fractals of the pentad Cantor sets. In order to investigate the thermodynamic stability of these pre-fractal domain structures, the electrostatic energy $U_E(n)$ and the wall energy $U_W(n)$ of the nth pre-fractal domain structure have been formulated as functions of the period in the framework of the Kittel theory. It is shown that under a constant period the electrostatic energy $U_E(3)$ of the third pre-fractal domain structure takes the least value while the wall energy $U_W(n)$ increases monotonically with increasing ordinal number n of the pre-fractal domain structure. The equilibrium half-period minimizing the internal energy including $U_E(n)$ and $U_W(n)$ is proportional to the square root of the thickness of a crystal plate. The equilibrium half-period increases monotonically with increasing ordinal number n of the pre-fractal domain structure. The equilibrium internal energy takes the least value in the zeroth pre-fractal domain structure and the second-least value in the second pre-fractal domain structure. The ratio of the latter to the former is nearly equal to unity, with the value of 1.10.

1. Introduction

Fractal aspects of macroscopic structures have been realized in various kinds of condensed matter [1]. Simple mechanisms dominating the formation of complex fractals have motivated many investigators. Recently fractal aspects have been discovered in the laminated ferroelectric 180° domain structure of potassium dihydrogen phosphate (KDP) [2]. Four types of self-similar domain pattern are optically observed to be periodic along the normal to the 180° walls in the stable state. Each half-period of the domain patterns shown in figure 1 is characterized by the first four generations of the pre-fractals of the pentad Cantor sets having the fractal dimension $D \approx 0.756$ along the domain wall normal. We call these patterns the pre-fractal domain patterns. The ordinal number n of the observed pre-fractal domain pattern increases without changing its period $2L$, as the numerical aperture of the objective lens of the polarizing optical microscope increases. However, an ultimate domain structure showing these pre-fractal domain patterns is not yet known. From a crystallographic standpoint, we can only make the prediction that the width of the narrowest domains in the nth pre-fractal domain structure, $\frac{1}{2}(\frac{2}{5})^n L$, would be slightly larger than the lattice spacing parallel to the domain wall normal, 14.852 Å at 90 K [3], in the case of the eighth pre-fractal domain structure having the period $2L = 9.1$ μm for example.

In the short-circuited KDP crystals in which spontaneous polarization charges are fully neutralized by true charges [4], the periods of the observed domain patterns are much

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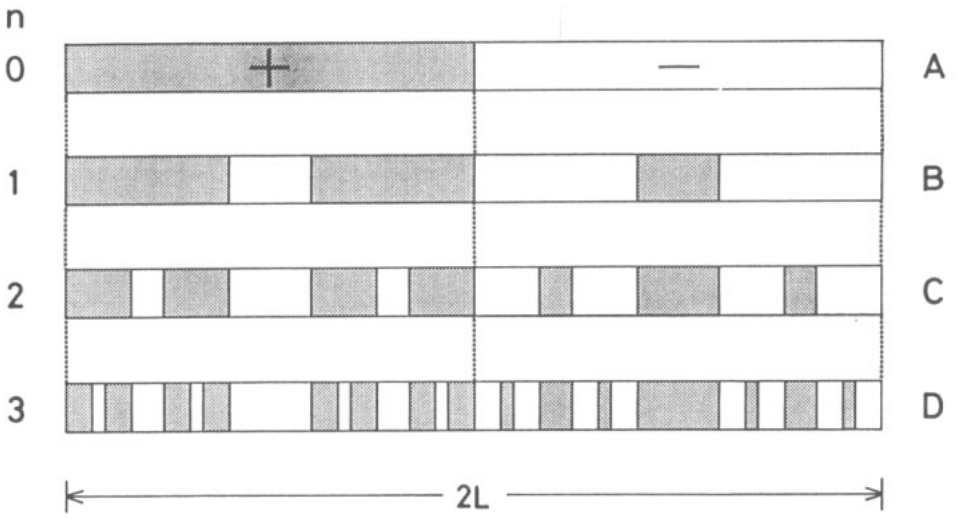

Figure 1. A schematic illustration of four pre-fractal domain structures having a period $2L$ along the normal to their 180° walls. The 180° walls are represented by the vertical solid lines in each nth pre-fractal domain structure. The spontaneous polarization vector is perpendicular to the surface of this page. The ordinal number n of the pre-fractal domain structure corresponds to the generation of a pre-fractal of the pentad Cantor sets having the fractal dimension $D \approx 0.756$ along the domain wall normal. The $(n+1)$th pre-fractal domain structure is obtained by reversing the polarity of a central fifth part of every + domain in the left half-period and the polarity of a central fifth part of every − domain in the right half-period in the nth pre-fractal domain structure. In the limit $n \to \infty$, the left half-period approaches the pentad Cantor sets having an infinite number of + domains with infinitesimal widths, while the right half-period the pentad Cantor sets having an infinite number of − domains with infinitesimal widths. A, B, C and D correspond to the pre-fractal domain patterns named in [2, 4]. A fractal dimension describing overall the period of these patterns is not yet known.

enlarged and their fractal aspects are lost owing to the absence of a depolarization field antiparallel to the spontaneous polarization vector. This verifies that the depolarization field is one of the essential factors stabilizing thermodynamically the pre-fractal domain structure of KDP. If we know the internal energy, including a contribution from the depolarization field generated in the nth pre-fractal domain structure, we shall be able to predict an ultimate domain structure at 0 K from a thermodynamic standpoint. Branched domain structure models [5] have already been proposed to explain how a demagnetization field causes ferromagnetic domains to branch near the surfaces of a uniaxial ferromagnetic plate. Unfortunately, however, no models have fractal aspects in domain branching.

In the present study, we take the simplest course in constructing the first model of the pre-fractal domain structures. The internal energy of the nth pre-fractal domain structure with the walls piercing through a uniaxial ferroelectric plate is formulated in the framework of the Kittel [6] theory. In the Kittel model, the 180° domain walls have no thicknesses but have a surface energy density. The domain width corresponding to the half-period of the zeroth pre-factal domain structure shown in figure 1 is diminished by the electrostatic energy originating from the depolarization field but is enlarged by the wall energy. The domain width minimizing the internal energy including the electrostatic energy and the wall energy is proportional to the square root of the thickness of a plate. The present pre-fractal domain structure model, which is an extension of the Kittel model, is a limiting case of a

branched domain structure model as far as the coercive field is concerned. The coercive field necessary to make domains branch is regarded as much weaker in the present model with the walls piercing through a plate than in a branched structure model with the walls localized near the surfaces of a plate. In the latter model, it is assumed that the coercive field is as strong as the electric field limiting the stability of a uniform ferroelectric phase described in the Landau phenomenological theory [5].

In order to extend the Kittel theory to the present case of the nth pre-fractal domain structure, we first derive the general formula for the electrostatic energy of a periodic 180° domain structure formed in a uniaxial ferroelectric plate as a function of the period $2L$ of the structure. Secondly we derive the formulae for the electrostatic energy $U_E(n)$ and the wall energy $U_W(n)$ of the nth pre-fractal domain structure. Thirdly we minimize the internal energy including $U_E(n)$ and $U_W(n)$ with respect to L. Finally we calculate numerically the equilibrium values of the internal energy and the period of the nth pre-fractal domain structure.

2. Internal energy

2.1. Electrostatic energy

As a first approach, we suppose a ferroelectric 180° domain structure formed in a semi-infinite crystal plate shown in figure 2 in the framework of the Kittel [6] theory. The domain structure has the period $2L$ along the x axis and the spontaneous polarization P_s along the z axis. Domain walls parallel to the y–z plane have no thicknesses. The charge density $\rho(x)$ per unit area on the surface $z = 0$ of the plate is an odd function of x and takes the value of either P_s or $-P_s$ irrespective of the passage of time. Here we assume that the domain structure shown in figure 2 is frozen at 0 K so that domain wall motion no longer takes place. Under the above condition, the relationship between the electric displacement and the electric field is regarded as approximately linear in the plate. Therefore, the electrostatic potential on the surface $z = 0$ can be obtained by solving the Laplace equations outside and inside the plate with consideration of the boundary conditions of an electric field (see appendix). Thus the general formula obtained for the electrostatic potential $\phi(x)$ on the surface $z = 0$ of the plate is expressed by

$$\phi(x) = \frac{2}{\pi\epsilon_0[1 + (\epsilon_x\epsilon_z)^{1/2}]} \sum_{m=1}^{\infty} \frac{1}{m} \left[\int_0^L \rho(x) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x \right] \sin\left(\frac{m\pi}{L}x\right). \tag{1}$$

Here ϵ_x and ϵ_z represent the principal dielectric constants of the plate along the x and the z axes, respectively. The factor $2/[1 + (\epsilon_x\epsilon_z)^{1/2}]$ amounts to the ϵ^* correction factor originating from the asymmetry of the electric field with respect to the surface $z = 0$ of the plate.

Substituting equation (1) into the relation

$$U_E = \frac{1}{2L} \int_0^L \rho(x)\phi(x)\,\mathrm{d}x$$

we obtain the general formula for the electrostatic energy U_E per unit area on the surface $z = 0$ of the plate:

$$U_E = \frac{1}{\pi\epsilon_0[1 + (\epsilon_x\epsilon_z)^{1/2}]L} \sum_{m=1}^{\infty} \frac{1}{m} \left[\int_0^L \rho(x) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x \right]^2. \tag{2}$$

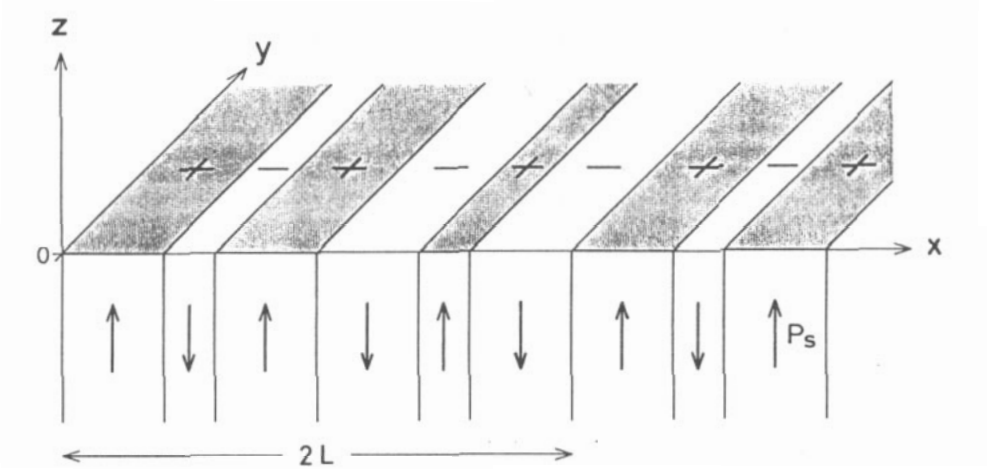

Figure 2. A schematic illustration of the ferroelectric 180° domain structure corresponding to the first pre-fractal domain structure in the semi-infinite crystal plate occupying the space $-\infty \leqslant x \leqslant \infty$, $-\infty \leqslant y \leqslant \infty$ and $z \leqslant 0$. The structure has the period $2L$ along the x axis and the spontaneous polarization P_s along the z axis. Domain walls parallel to the y–z plane have no thicknesses but have the surface energy density σ.

Calculating the integration in equation (2) in the nth pre-fractal domain structure shown in figure 1, we obtain the electrostatic energy $U_E(n)$ of the nth pre-fractal domain structure:

$$U_E(n) = \frac{4P_s^2 L}{\pi^3 \epsilon_0 [1 + (\epsilon_x \epsilon_z)^{1/2}]} \alpha(n) \tag{3}$$

where

$$\alpha(0) = \sum_{m=1}^{\infty} \frac{1}{m^3} \{\tfrac{1}{2}[1 - \cos(m\pi)]\}^2 \equiv \sum_{m=1}^{\infty} \frac{1}{m^3} \beta_0(m)^2 \simeq 1.052 \tag{4}$$

$$\alpha(1) = \sum_{m=1}^{\infty} \frac{1}{m^3} \left[\beta_0(m) + \cos\left(\frac{3m\pi}{5}\right) - \cos\left(\frac{2m\pi}{5}\right)\right]^2 \equiv \sum_{m=1}^{\infty} \frac{1}{m^3} \beta_1(m)^2 \simeq 0.435$$

$$\alpha(2) = \sum_{m=1}^{\infty} \frac{1}{m^3} \left[\beta_1(m) + \cos\left(\frac{6m\pi}{25}\right) - \cos\left(\frac{4m\pi}{25}\right) + \cos\left(\frac{21m\pi}{25}\right) - \cos\left(\frac{19m\pi}{25}\right)\right]^2$$
$$\equiv \sum_{m=1}^{\infty} \frac{1}{m^3} \beta_2(m)^2 \simeq 0.183$$

$$\alpha(3) = \sum_{m=1}^{\infty} \frac{1}{m^3} \left[\beta_2(m) + \cos\left(\frac{12m\pi}{125}\right) - \cos\left(\frac{8m\pi}{125}\right) + \cos\left(\frac{42m\pi}{125}\right) - \cos\left(\frac{38m\pi}{125}\right)\right.$$
$$\left. + \cos\left(\frac{87m\pi}{125}\right) - \cos\left(\frac{83m\pi}{125}\right) + \cos\left(\frac{117m\pi}{125}\right) - \cos\left(\frac{113m\pi}{125}\right)\right]^2$$
$$\equiv \sum_{m=1}^{\infty} \frac{1}{m^3} \beta_3(m)^2 \simeq 0.117$$

It is difficult to express $\alpha(n)$, the sum of series defined in equation (3), by an explicit function of the ordinal number n of the pre-fractal domain structure. The electrostatic energy $U_E(0)$ of the zeroth pre-fractal domain structure, which is obtained by substituting equation (4) into equation (3), coincides with the formula reported previously [6, 7].

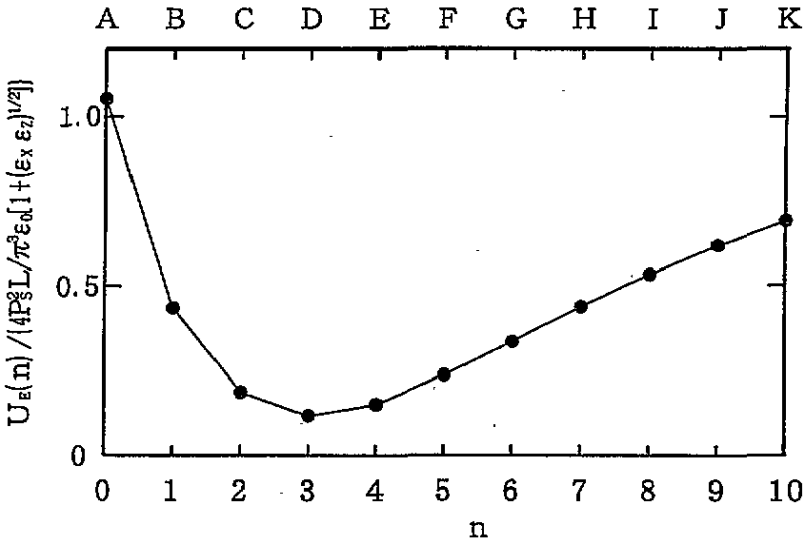

Figure 3. The normalized value of the electrostatic energy $U_E(n)$ of the nth pre-fractal domain structure characterized by the nth pre-fractal of the pentad Cantor sets in each half-period. The normalized $U_E(n)$ is equivalent to $\alpha(n)$, the sum of series defined by equation (3).

The value of the normalized $U_E(n)$, which is equivalent to $\alpha(n)$ in equation (3), was calculated numerically and is shown in figure 3. The figure shows that the electrostatic energy $U_E(n)$ takes the least value in the third pre-fractal domain structure under a constant period $2L$. The third pre-fractal domain structure shown in figure 1 has the volume ratio of $+$ domains to $-$ domains most nearly equal to unity: 1.05 in the left half-period and 0.95 in the right half-period.

Figure 3 suggests that the electrostatic energy $U_E(n)$ gradually approaches $U_E(\infty)$ for the hypothetical fractal domain structure corresponding to the pentad Cantor sets in each half-period. The calculated ratio $U_E(17)/U_E(0) \simeq 0.925$ predicts that the value of $U_E(\infty)$ will be equal to that of $U_E(0)$.

2.2. Domain wall energy

We suppose that the crystal plate shown in figure 2 has the thickness d along the z axis and every domain wall has the surface energy σ per unit area. Then, the domain wall energy $U_W(n)$ of the nth pre-fractal domain structure in a pillar having the cross section of a unit area on both surfaces of the plate is proportional to the density of walls $(2^{n+1}-1)/L$:

$$U_W(n) = (2^{n+1}-1)\frac{\sigma d}{L}. \tag{5}$$

The wall energy $U_W(0)$ of the zeroth pre-fractal domain structure, which is given by equation (5), coincides with the formula reported previously [6, 7]. The wall energy $U_W(n)$ of the nth pre-fractal domain structure increases monotonically with increasing generation n under a constant period $2L$.

2.3. Internal energy including electrostatic energy and wall energy

We can neglect the electrostatic interaction between both surfaces of the plate in which the domains are very thin, $d \gg L$ [7]. Then, the internal energy $U(n)$ of the nth pre-fractal domain structure in a pillar having the cross section of a unit area on both surfaces of the plate is given by the sum of twice the electrostatic energy $U_E(n)$ expressed by equation (3) and the wall energy $U_W(n)$ expressed by equation (5):

$$U(n) = \frac{8P_s^2 L\alpha(n)}{\pi^3\epsilon_0[1+(\epsilon_x\epsilon_z)^{1/2}]} + (2^{n+1}-1)\frac{\sigma d}{L}. \tag{6}$$

The half-period $L_{eq}(n)$ minimizing the energy $U(n)$ is obtained using the equilibrium condition $dU(n)/dL = 0$, as follows:

$$L_{eq}(n) = \left(\frac{\pi^3\epsilon_0[1+(\epsilon_x\epsilon_z)^{1/2}]\sigma d(2^{n+1}-1)}{8P_s^2\alpha(n)}\right)^{1/2}. \tag{7}$$

The half-period $L_{eq}(0)$ given by equation (7), which corresponds to the domain width of the zeroth pre-fractal domain structure shown in figure 1, coincides with the formula reported previously [7]. The most important point in equation (7) is that the equilibrium half-period $L_{eq}(n)$ of the nth pre-fractal domain structure is proportional to the square root of the thickness of the plate d. This explains the thickness dependence of the half-period of the pre-fractal domain patterns obtained in unelectroded KDP c plates [8]. Substituting equation (7) into equation (6), we obtain the equilibrium internal energy $U_{eq}(n)$ of the nth pre-fractal domain structure:

$$U_{eq}(n) = 2\left(\frac{8P_s^2\sigma d\alpha(n)(2^{n+1}-1)}{\pi^3\epsilon_0[1+(\epsilon_x\epsilon_z)^{1/2}]}\right)^{1/2}. \tag{8}$$

The values of the normalized $L_{eq}(n)$ and $U_{eq}(n)$, which are equivalent to $[(2^{n+1}-1)/\alpha(n)]^{1/2}$ in equation (7) and $[\alpha(n)(2^{n+1}-1)]^{1/2}$ in equation (8), respectively, were calculated numerically and are shown in figure 4. The equilibrium half-period $L_{eq}(n)$ increases monotonically with increasing n, while the equilibrium internal energy $U_{eq}(n)$ takes the least value in the zeroth pre-fractal domain structure and the second-least value in the second pre-fractal domain structure. The ratio of the two values is nearly equal to unity: $U_{eq}(2)/U_{eq}(0) \simeq 1.10$.

Figure 4 suggests that the hypothetical fractal domain structure has infinite values of $L_{eq}(\infty)$ and $U_{eq}(\infty)$.

3. Discussion

The internal energy including only the electrostatic energy $U_E(n)$ shows, in figure 3, that the third pre-fractal domain structure is stable under a constant period. This seems to support the fact that pre-fractal domain patterns having an ordinal number larger than 3 have not yet been observed. However, we cannot conclude that the third pre-fractal domain structure is the ultimate irrespective of its period, because it is necessary for us to observe narrow domains having widths less than the optical resolving power from now on. The thickness dependence of the half-period obtained for KDP [8] suggests the importance of the wall energy $U_W(n)$ as well as the electrostatic energy $U_E(n)$. The internal energy including $U_E(n)$ and $U_W(n)$

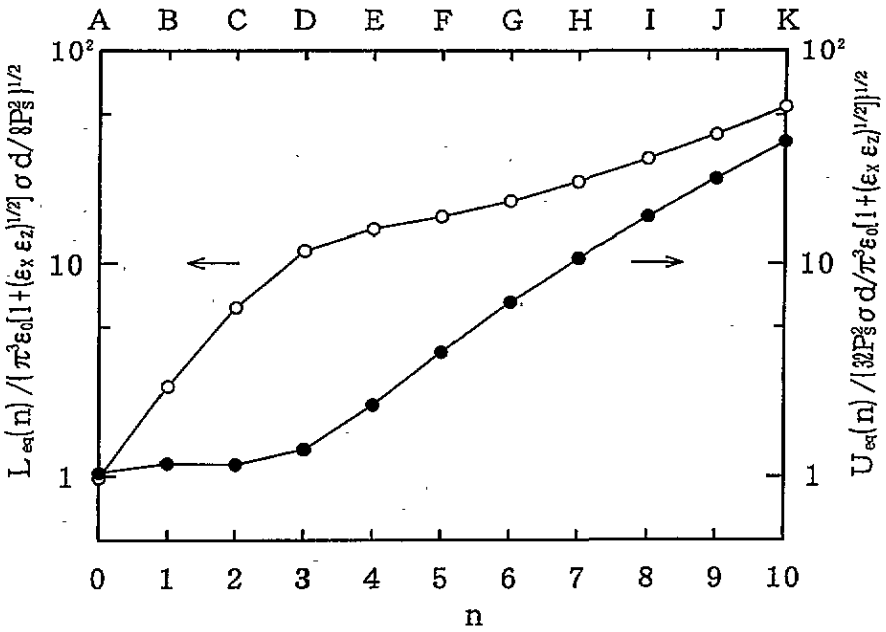

Figure 4. Normalized values of the equilibrium internal energy $U_{eq}(n)$ and the equilibrium half-period $L_{eq}(n)$ of the nth pre-fractal domain structure characterized by the nth pre-fractal of the pentad Cantor sets in each half-period. The normalized $U_{eq}(n)$ and $L_{eq}(n)$ are equivalent to $[\alpha(n)(2^{n+1}-1)]^{1/2}$ in equation (8) and $[(2^{n+1}-1)/\alpha(n)]^{1/2}$ in equation (7), respectively.

shows, in figure 4, that the zeroth pre-fractal domain structure is stable and the second pre-fractal domain structure is metastable. This contradicts the observations that the pre-fractal domain patterns having $n = 1$, 2 and 3, as well as the zeroth pattern, are observed in the stable state. We can think of two ways to improve the present pre-fractal domain structure model constructed in the framework of the Kittel model. The first way is to reconstruct the pre-fractal domain structure model on the basis of a branched domain structure model. It is expected that the zeroth pre-fractal domain structure would become unstable in plates having large thicknesses so that branched domain structures would become localized near the surfaces of the plates [5]. The second way is to introduce a contribution of entropy into the pre-fractal domain structure model. Fortunately, the equilibrium values $U_{eq}(1)$, $U_{eq}(2)$ and $U_{eq}(3)$ of the internal energy are only slightly larger than that of $U_{eq}(0)$. The pre-fractal domain structures having $n = 1$, 2 and 3 can become stable at high temperatures by a small contribution of entropy.

Physical properties characteristic of KDP are not taken into consideration in the present calculation of $U_{eq}(n)$. Measurement of x-ray diffuse scattering in KDP has clarified that the domain wall has a statically modulated structure with the pattern of the atomic displacements similar to those occurring in ferroelectric fluctuations [9]. This suggests that each domain wall itself has entropy. If the domain wall entropy $S(n)$ of the nth pre-fractal domain structure increases with increasing ordinal number n, the free energy of the nth pre-fractal domain structure $F(n) = U_E(n) + U_W(n) - TS(n)$ will take the least value in an nth generation other than the zeroth generation at high temperatures.

The internal energy of the hypothetical fractal domain structure corresponding to the pentad Cantor sets in each half-period should be interesting from a mathematical point

of view. In figure 1, we can easily imagine the pre-fractal domain structure in the limit $n \to \infty$, namely the fractal domain structure. The fractal domain structure has an infinite number of + domains having infinitesimal widths in the left half-period and an infinite number of − domains having infinitesimal widths in the right half-period. The + domains have no volume fraction in the left half-period and the − domains have no volume fraction in the right half-period. As a result, the fractal domain structure takes the same value of the integration in equation (2) as the zeroth pre-fractal domain structure having a − domain in the left half-period and a + domain in the right half-period does. Therefore, the sum of series $\alpha(\infty)$ of the fractal domain structure takes the finite value equal to $\alpha(0) \simeq 1.052$ of the zeroth pre-fractal domain structure presented in equation (4). This argument is supported by figure 3, indicating that $U_E(n) \to U_E(0)$ in the limit $n \to \infty$. On the other hand, the fractal domain structure has an infiite value of the domain wall energy $U_W(\infty)$ in equation (5) because of an infinite number of domain walls in each half-period. Finally, it is concluded that both the equilibrium internal energy $U_{eq}(\infty)$ given by equation (8) and the equilibrium half-period $L_{eq}(\infty)$ given by equation (7) have infinite values. Figure 4 indicates these infinite values. We cannot introduce the domain wall entropy into the fractal domain structure having an infinite number of domain walls with no thicknesses. Therefore, the hypothetical fractal domain structure would not become stable at any temperature.

It is interesting to know why KDP has the 180° domain structure characterized not by pre-fractals of the well known 'triadic' Cantor sets [1] but by the pre-fractals of the present 'pentad' Cantor sets. The internal energy of 180° domain structures characterized by pre-fractals of other kinds of Cantor set will be published elsewhere to clarify the outstanding aspects of the present pre-fractal domain structures characterized by the pentad Cantor sets.

Acknowledgment

The authors are grateful to Professor E Nakamura for his encouragement.

Appendix. Derivation of the electrostatic potential

The general formula (1) for the electrostatic potential on the surface $z = 0$ of the periodic domain structure shown in figure 2 is derived in the following two stages. In the first stage, we derive the electrostatic potential on the dielectrically isotropic plate where the principal dielectric constants along the x and the z axes are $\epsilon_x = \epsilon_z = 1$. In the second stage, we derive the electrostatic potential on the dielectrically anisotropic plate where $\epsilon_x \neq \epsilon_z \neq 1$ by applying the ϵ^* correction, the dielectric version of the μ^* correction in ferromagnets [6], on the potential derived in the first stage.

In the first stage where $\epsilon_x = \epsilon_z = 1$, the electrostatic potential $\phi_1(x, z)$ satisfies the Laplace equation

$$\Delta\phi_1(x, z) \equiv \phi_{1xx}(x, z) + \phi_{1zz}(x, z) = 0 \tag{A1}$$

except on the surface $z = 0$ of the plate shown in figure 2. Solving equation (A1) under the boundary conditions $\phi_1(0, z) = \phi_1(L, z) = 0$ and $\phi_1(x, -\infty) = \phi_1(x, +\infty) = 0$, we obtain the well-known solution

$$\phi_1(x, z) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi}{L}x\right) \exp\left(-\frac{m\pi}{L}|z|\right). \tag{A2}$$

The z component $E_z = -\phi_{1z}$ of the electric field satisfies the boundary condition on the surface $z = 0$:

$$\phi_{1z}(x, 0+) - \phi_{1z}(x, 0-) = -\rho(x)/\epsilon_0. \tag{A3}$$

The symmetry of the field E_z with respect to the surface $z = 0$ is expressed by

$$\phi_{1z}(x, 0+) = -\phi_{1z}(x, 0-). \tag{A4}$$

Equation (A3) is rewritten using equation (A4), as follows:

$$\phi_{1z}(x, 0-) = \rho(x)/2\epsilon_0. \tag{A5}$$

Substituting $\phi_{1z}(x, 0)$ derived from equation (A2) into equation (A5), we obtain the Fourier sine series expansion for the odd function $\rho(x)$:

$$\rho(x) = \frac{2\pi\epsilon_0}{L} \sum_{m=1}^{\infty} C_m m \sin\left(\frac{m\pi}{L}x\right). \tag{A6}$$

Equation (A6) gives the coefficient C_m in equation (A2):

$$C_m = \frac{1}{\pi\epsilon_0 m} \int_0^L \rho(x) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x. \tag{A7}$$

Thus we derive the electrostatic potential on the surface $z = 0$ in the first stage where $\epsilon_x = \epsilon_z = 1$:

$$\phi_1(x, 0) = \frac{1}{\pi\epsilon_0} \sum_{m=1}^{\infty} \frac{1}{m} \left[\int_0^L \rho(x) \sin\left(\frac{m\pi}{L}x\right) \mathrm{d}x\right] \sin\left(\frac{m\pi}{L}x\right). \tag{A8}$$

In the second stage where $\epsilon_x \neq \epsilon_z \neq 1$, we suppose that the electrostatic potential $\phi_2(x, z)$ is expressed by scaling the harmonic function $\phi_1(x, z)$ along the z axis, as follows:

$$\phi_2(x, z) = \begin{cases} A\phi_1(x, \zeta z) & \text{for } z \geqslant 0 \quad \text{(A9)} \\ A\phi_1(x, \xi z) & \text{for } z \leqslant 0. \quad \text{(A10)} \end{cases}$$

Here A is called the ϵ^* correction factor, and ζ and ξ are the scaling factors. In contrast with the first stage, the z component of the electric field $E_z = -\phi_{2z}$ is asymmetric with respect to the surface $z = 0$ and satisfies the boundary condition

$$\epsilon_0\phi_{2z}(x, 0+) - \epsilon_z\epsilon_0\phi_{2z}(x, 0-) = -\rho(x). \tag{A11}$$

Substituting equations (A9) and (A10) into equation (A11) and using the relations (A4) and (A5), we obtain the following relation between the factors:

$$A(\zeta + \epsilon_z\xi) = 2. \tag{A12}$$

For $z > 0$, the potential $\phi_2(x, z)$ satisfies the Laplace equation

$$\Delta\phi_2 \equiv \phi_{2xx}(x, z) + \phi_{2zz}(x, z) = 0. \tag{A13}$$

Substituting equation (A9) into equation (A13) and comparing with the Laplace equation (A1), we obtain one of the scaling factors:

$$\zeta = 1. \tag{A14}$$

For $z < 0$, on the other hand, the potential $\phi_2(x, z)$ satisfies the Laplace equation in the plate:

$$\epsilon_x \phi_{2xx}(x, z) + \epsilon_z \phi_{2zz}(x, z) = 0. \tag{A15}$$

Substituting equation (A10) into equation (A15) and using the Laplace equation (A1) valid at any point except $z = 0$, we obtain the other scaling factor:

$$\xi = (\epsilon_x/\epsilon_z)^{1/2}. \tag{A16}$$

Substituting equations (A14) and (A16) into equation (A12), we obtain the ϵ^* correction factor:

$$A = 2/[1 + (\epsilon_x \epsilon_z)^{1/2}]. \tag{A17}$$

After substituting equations (A8) and (A17) into either equation (A9) or (A10), we finally obtain the general formula (1) for the electrostatic potential on the surface $z = 0$ of the periodic domain structure shown in figure 2.

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